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# Technical Notes

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## Smoothing of the Multiple One-Dimensional Adaptive Grid Procedure

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NUMERICAL solutions of partial differential equations are strongly dependent upon the grid distributions. It is well known that, for both the structured and unstructured grids, if the grid smoothness can be maintained to a certain degree in a structured grid system the accuracy of the solution is determined by the grid adaptivity to the variation of physical variables.[1–3] Although the grid adaptation of the structured grids is worse than that of the unstructured grids, the present study focuses on the former. At the present, adaptive grid methods can roughly be divided into three classes: variational method, method of solving elliptical equation, and equidistribution method. The variational method resolves both the grid adaptivity and grid smoothness very well. The method of solving elliptical equation employs grid-control functions to handle the adaptation and provides the grid smoothness by the averaging procedure of solving the elliptical equation. If the control function is properly designed, the resulting grid adaptivity and quality are comparable to the variational method. It can be easily proved that the equidistribution method is a one-dimensional variational method.[2,4,5] The one-dimensional equidistribution method is first extended to a multidimensional problem by Shyy.[6] For a multidimensional problem the resulting grid adaptivity and grid quality of a multiple one-dimensional adaptive grid procedure are generally worse than the corresponding ones of the variational and elliptical equation methods. However, the variational and elliptical equation methods generally require much longer computing time than the equidistribution method does.

Shyy[6] applied the one-dimensional grid adjustment procedure of Dwyer et al.[4,5] to those grid lines where $\xi$ = constant and then to lines where $\eta$ = constant or vice versa. Jeng and Liou[7] and Jeng and Lee[8] modified the weighting function of the one-dimensional adaptive grid procedure by coupling information from neighboring points. However, grid skewness around an isolated high gradient region remains a problem for the multiple one-dimensional adaptive method. The present study extends the two-dimensional smoothing method of Ref. 9 to the multiple one-dimensional adaptive grid procedure.

### Theoretical Analysis

Suppose that an initial solution has already been obtained on an initial two-dimensional structured grid system. For convenience, grid lines are denoted as $\xi$ and $\eta$ = constant lines. The one-dimensional equidistribution method of Dwyer et al.[4,5] employs the concept of equidistributing weight functions along a grid line in the following form:

$$W_i \Delta s_i = 1 + \lambda \left|\frac{\partial T}{\partial s}\right|_i \Delta s_i = \text{constant} \tag{1}$$

$$m_i = m \frac{\sum_{i=0}^{i} 1 + \lambda|\Delta T/\Delta s|_{i,j}\Delta s_{i,j}}{\sum_{i=0}^{i_{\max}} 1 + \lambda|\Delta T/\Delta s|_{i,j}\Delta s_{i,j}} \tag{2}$$

where

$$\Delta T_{i,j} = T_{i,j} - T_{i-1,j}$$

$$\Delta s_{i,j} = \sqrt{(x_{i,j} - x_{i-1,j})^2 + (y_{i,j} - y_{i-1,j})^2}$$

$T$ denotes a physical variable, $\lambda$ is the adaptive factor, and $m$ is the total number of grids along a $j$ = constant line. Suppose that within a line segment $\Delta s_m$ of the old grid system where $|\Delta T/\Delta s|_{i,j}$ attains a maximum value, and we want to distribute $\Delta m_i$ points approximately. The adaptive factor can then be estimated as

$$\lambda = \frac{\Delta m_i - 1}{|\Delta T|_{\max} - (\Delta m_i/m)\sum_{i=0}^{i_{\max}} |\Delta T|_{i,j}} \tag{3}$$

where $|\Delta T|_{\max} = |\Delta T/\Delta s|_{\max}\Delta s_m$. Because the parameter $\lambda$ should be positive, the prescribed $\Delta m_i$ should satisfy

$$1 \le \Delta m_i \le \frac{m|\Delta T|_{\max}}{\sum_{i=0}^{i_m} |\Delta T|_{i,j}} \tag{4}$$

The following constraint is proposed to avoid a situation that the term on the far right of the preceding inequality may be too close to unity:

$$1/\Delta s_{i,j} + \lambda|\Delta T/\Delta s|_{i,j} \le 1/\Delta s_{\min} \tag{5}$$

where $\Delta s_{\min}$ is an user-specified parameter. By using Eq. (2) as a base, the approximate grid stretching between two successive grid segments and the ratio between the largest and smallest grid size can be estimated by, respectively,

$$\frac{\Delta s_{i+1,j}}{\Delta m_{i+1}} \Bigg/ \frac{\Delta s_{i,j}}{\Delta m_i} \sim \frac{\Delta s_{i+1,j}}{\Delta s_{i,j}} \frac{1 + \lambda|\Delta T|_{i,j}}{1 + \lambda|\Delta T|_{i+1,j}} \tag{6}$$

$$\frac{\Delta s^{\text{new}}_{\max}}{\Delta s^{\text{new}}_{\min}} \sim \frac{\Delta s^{\text{old}}_{\min}}{\Delta s^{\text{old}}_{\max}} \frac{1 + \lambda \Delta T_{\max}}{1 + \lambda \Delta T_{\min}} \tag{7}$$

in which the subscript min denotes the minimum $|\Delta T/\Delta s|$, whereas the subscript max denotes the maximum $|\Delta T/\Delta s|$.

If the flowfield is complicated and variations of dependent variables are significant, numerical experiments show that the improved method of Refs. 7 and 8 for the multiple one-dimensional adaptive grid procedure cannot effectively eliminate grid skewness, and the resulting grids can lead to an unstable calculation of the flow solver. A careful examination of Eqs. (2), (6), and (7) and numerical experiments reveals that the difference of grid distributions along two adjacent grid lines of the first sweeping direction dominates the grid skewness. The difference of the second sweeping direction is much less important than that of the first sweeping. Moreover, the difference of grid distributions between two adjacent grid lines is introduced by the difference of $W_{i,j}(s)$ along these lines. Based on these

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discussions, the Newton–Cotes integration formula is employed to smooth the $|\partial T/\partial s|$ term as follows[9]:

$$W_{i,j}(s) = \frac{1}{\Delta s_{i,j}} + \lambda \left| \frac{\partial T}{\partial s} \right|_{i,j}$$

$$\left| \frac{\partial T}{\partial s} \right|_{i,j} = \left[ (1-\omega) \left| \frac{\partial T}{\partial s} \right|_{i,j}^{\text{old}} + \omega \sum_{\ell=-n}^{n} \sum_{k=-n}^{n} \alpha_{\ell,k} \left| \frac{\partial T}{\partial s} \right|_{i+\ell,j+k}^{\text{old}} \right] \tag{8}$$

where $\alpha_{i,j}$ are the coefficients of the Newton–Cotes integration formula. To avoid excessive grid stretching along a grid line, the following smoothing between successive cells is employed once the stretching ratio $\Delta s^{\text{new}}_{i,j}/\Delta s^{\text{new}}_{i+1,j}$ is larger than $r$ or less than $1/r$, where $r \approx 1.3$ is employed:

$$\Delta T_{i,j} = (1-\omega_\ell)\Delta T^{\text{old}}_{i,j} + \omega_\ell \Delta T^{\text{old}}_{i+1,j}$$

$$\Delta T_{i+1,j} = (1-\omega_\ell)\Delta T^{\text{old}}_{i+1,j} + \omega_\ell \Delta T^{\text{old}}_{i,j} \tag{9}$$

Reduction of the difference between the grid distributions along two adjacent lines is to be obtained by the following scheme. By considering the $i$th points on $j$th and $(j+1)$th lines, the arclengths of these points in the resulting grids after an adaptive sweeping are $s^{\text{new}}_{i,j}$ and $s^{\text{new}}_{i,j+1}$, respectively, where $s^{\text{old}}_{k,j} \le s^{\text{new}}_{i,j} \le s^{\text{old}}_{k+1,j}$ and $s^{\text{old}}_{\ell,j+1} \le s^{\text{new}}_{i,j+1} \le s^{\text{old}}_{\ell+1,j+1}$. For convenience the $(i, j)$ point is projected onto the $(j+1)$th line. Assume that grid smoothness is preserved if the following criterion is satisfied:

$$\Delta s_{i,j} = \left| s^*_{i,j+1} - s^{\text{new}}_{i,j+1} \right| \le \epsilon \left( s^{\text{old}}_{k+1,j+1} - s^{\text{old}}_{k,j+1} \right) \tag{10}$$

where

$$s^*_{i,j+1} = s^{\text{old}}_{k,j+1} + \left( s^{\text{old}}_{k+1,j+1} - s^{\text{old}}_{k,j+1} \right) \frac{s^{\text{new}}_{i,j} - s^{\text{old}}_{k,j}}{s^{\text{old}}_{k+1,j} - s^{\text{old}}_{k,j}}$$

$s^*_{i,j+1}$ is the projection of $s^{\text{new}}_{i,j}$ on the $(j+1)$th line, and the parameter $\epsilon$ takes the value of 0.25–0.3 in this study. Subsequently, the following across-line smoothing is performed once the inequality of Eq. (10) is violated:

$$\sum_{\ell=1}^{i} \left| \frac{\Delta T}{\Delta s} \right|_{\ell,j} \Delta s_{\ell,j} = \omega_1 \sum_{\ell=1}^{i} \left| \frac{\Delta T}{\Delta s} \right|_{\ell,j-1} \Delta s_{\ell,j-1} + (1-\omega_1-\omega_2) \sum_{\ell=1}^{i} \left| \frac{\Delta T}{\Delta s} \right|_{\ell,j} \Delta s_{\ell,j} + \omega_2 \sum_{\ell=1}^{i} \left| \frac{\Delta T}{\Delta s} \right|_{\ell,j+1} \Delta s_{\ell,j+1} \tag{11}$$

## Results and Discussion

A two-dimensional inviscid oblique shock reflection problem is considered. An inviscid supersonic inflow of $M_{\text{inlet}} = 1.4$ runs across a channel with a 4% circular bump. An initial grid system of $67 \times 35$ points was generated by distributing vertical grid lines in equal spacing along the $x$ direction. Subsequently, uniform grids are given along the vertical grid lines. The initial grids are not shown here because of length limitation. The Yee–Harten minmod total-variation-diminishing (TVD) scheme[10] is employed to provide an initial solution that is shown in Fig. 1. With Fig. 1 as a base if the improved multiple one-dimensional adaptive grid procedure of Jeng and Lee[8] is employed, grid oscillations cannot be avoided in the resulting adaptive grids, and a convergent physical solution cannot be obtained.

Numerical tests show that the following combinations give nearly the same results: applying two-dimensional smoothing three times, applying two-dimensional smoothing two times followed by smoothing between cells and across-line smoothing one time, and employing two-dimensional smoothing one time plus smoothing between cells five times and then across-line smoothing one time,

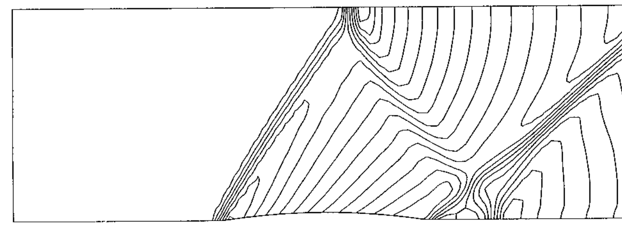

**Fig. 1 TVD solution based on the initial coarse grids.**

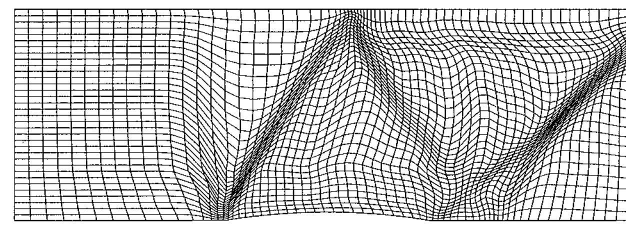

**Fig. 2 Adaptive grid system generated by applying two-dimensional smoothing once followed by applying the smoothing between successive cells five times ($\omega_\ell = 0.25, 4/3 > \Delta s^{\text{new}}_{i,j}/\Delta s^{\text{new}}_{i+1,j} > 3/4$) and smoothing across lines ($\omega_1 = \omega_2 = 0.25$, one time), respectively.**

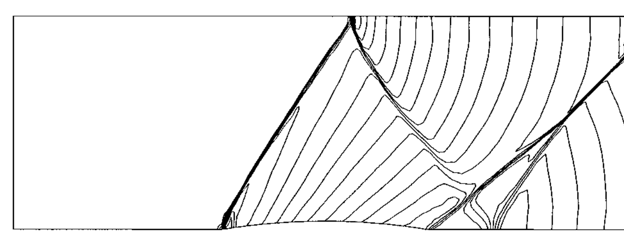

**Fig. 3 TVD solution based on the adaptive grids of Fig. 2.**

where $\omega = 3$, $\omega_\ell = \omega_1 = \omega_2 = 0.25$. If the parameters are moderately varied ($1 \le \omega \le 5$ and $0.1 \le \text{other } \omega \le 0.5$), the grid adaptation is not significantly changed. Those shown in Figs. 2 and 3 are the result of the last combination, which is slightly better than the other two combinations. The resulting solution shown in Fig. 3 resolves the reflected oblique shocks much better than the initial solution of Fig. 1 does. Although the solution of the present approach is slightly worse than that of the elliptical equation method (Fig. 10 of Ref. 11), its simplicity and computation efficiency are attractive.

## Conclusions

The fast multiple one-dimensional adaptive grid procedure is modified and is applied to a two-dimensional supersonic inviscid oblique shock-wave reflection problem. The modifications include three smoothing techniques: two-dimensional smoothing, smoothing between successive cells, and across-line smoothing. Numerical experiments of the test case show that these three techniques can be arranged to obtain satisfactory grid adaptivity.

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# Derivatives of Complex Eigenvectors Using Nelson's Method

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## Nomenclature

| | | |
|---|---|---|
| $c_i, d_i$ | = | scalars used to calculate the eigenvector derivatives |
| $\boldsymbol{M}, \boldsymbol{D}, \boldsymbol{K}$ | = | mass, damping, and stiffness matrices |
| $s_i$ | = | system eigenvalues |
| $\boldsymbol{u}_i, \boldsymbol{v}_i$ | = | left and right system eigenvectors |
| $\boldsymbol{x}_i, \boldsymbol{y}_i$ | = | vectors used to calculate the eigenvector derivatives |
| $\theta$ | = | system parameter |
| $\{\ \}_e$ | = | $e$th element of a vector |

## Introduction

THE calculation of derivatives of natural frequencies and mode shapes with respect to model parameters is vital for design optimization, model updating, fault detection, and many other applications.[1,2] The methods to calculate these derivatives are well established for undamped structures. Fox and Kapoor[3] calculated the derivative of the eigenvectors by expressing these derivatives as a linear combination of the undamped eigenvectors. Nelson[4] introduced the approach, extended in this Note, where only the eigenvector of interest was required. Adhikari[1] extended the method of Fox and Kapoor to systems with nonproportional damping using the mass, damping, and stiffness matrices directly. Lee et al.[5] calculated the eigenvector derivatives of self-adjoint systems using a similar approach to Nelson. This Note extends Nelson's method to nonproportionally damped systems with complex modes. This method has the great advantage that only the eigenvector of interest is required. The method proposed by Adhikari[1] obtained the eigenvector derivative as a linear combination of all of the eigenvectors. For large-scale structures, with many degrees of freedom, obtaining all of the eigenvectors is a computationally expensive task. Both self-adjoint and non-self-adjoint systems are considered.

The eigenvalues and corresponding right and left eigenvectors of the standard equations of motion in structural dynamics in second-order form are given by the solutions of

$$\left(s_i^2\boldsymbol{M} + s_i\boldsymbol{D} + \boldsymbol{K}\right)\boldsymbol{u}_i = 0 \tag{1}$$

$$\boldsymbol{v}_i^T\left(s_i^2\boldsymbol{M} + s_i\boldsymbol{D} + \boldsymbol{K}\right) = 0 \tag{2}$$

Often the structural matrices are symmetric, but here we allow the possibility that the matrices are asymmetric. For the self-adjoint case (symmetric matrices) the left and right eigenvectors are equal, $\boldsymbol{u}_i = \boldsymbol{v}_i$. Also, the eigenvalues and eigenvectors must occur in complex conjugate pairs because the structural matrices are real. Furthermore, in this Note we assume that the eigenvalues are distinct.

The eigenvectors are not unique in the sense that any scalar (complex) multiple of an eigenvector is also an eigenvector. There are numerous ways of introducing a normalization to ensure uniqueness. For undamped systems mass normalization is the most common. A useful normalization for damped systems is

$$\boldsymbol{v}_i^T[s_i\boldsymbol{M} + (1/s_i)\boldsymbol{K}]\boldsymbol{u}_i = \boldsymbol{v}_i^T(2s_i\boldsymbol{M} + \boldsymbol{D})\boldsymbol{u}_i = 1 \tag{3}$$

which ensures that the eigenvectors have the equivalent scaling to the measured eigenvectors.[6,7] One disadvantage of this scaling is that "real" modes, produced by proportionally damped models, are multiplied by a complex scalar.

## Eigenvalue Derivatives

Adhikari[1] obtained the derivatives of the eigenvalues for the self-adjoint case. Here we extend the approach to the general case. Differentiating Eq. (1) with respect to the parameter $\theta$ gives

$$\left(s_i^2\frac{\partial\boldsymbol{M}}{\partial\theta} + s_i\frac{\partial\boldsymbol{D}}{\partial\theta} + \frac{\partial\boldsymbol{K}}{\partial\theta}\right)\boldsymbol{u}_i + (2s_i\boldsymbol{M} + \boldsymbol{D})\boldsymbol{u}_i\frac{\partial s_i}{\partial\theta} + \left(s_i^2\boldsymbol{M} + s_i\boldsymbol{D} + \boldsymbol{K}\right)\frac{\partial\boldsymbol{u}_i}{\partial\theta} = 0 \tag{4}$$

Equation (4) is now premultiplied by $\boldsymbol{v}_i^T$. The third term is then zero from Eq. (2), and the scaling makes the coefficient of the eigenvalue derivative unity. Thus

$$\frac{\partial s_i}{\partial\theta} = -\boldsymbol{v}_i^T\left(s_i^2\frac{\partial\boldsymbol{M}}{\partial\theta} + s_i\frac{\partial\boldsymbol{D}}{\partial\theta} + \frac{\partial\boldsymbol{K}}{\partial\theta}\right)\boldsymbol{u}_i \tag{5}$$

## Eigenvector Derivatives: Self-Adjoint Case

In the self-adjoint case the left and right eigenvectors are equal, and the eigenvector derivative satisfies, from Eq. (4),

$$\left(s_i^2\boldsymbol{M} + s_i\boldsymbol{D} + \boldsymbol{K}\right)\frac{\partial\boldsymbol{u}_i}{\partial\theta} = \boldsymbol{h}_i \tag{6}$$

where the vector $\boldsymbol{h}_i$ consists of the first two terms in Eq. (4) and all of these quantities are now known. Equation (6) cannot be solved to obtain the eigenvector derivative because the matrix is singular. For distinct eigenvalues this matrix has a null space of dimension one. Following Nelson's approach, the eigenvector derivative is written as

$$\frac{\partial\boldsymbol{u}_i}{\partial\theta} = \boldsymbol{x}_i + c_i\boldsymbol{u}_i \tag{7}$$

where $\boldsymbol{x}_i$ and $c_i$ have to be determined. These quantities are not unique because any multiple of the eigenvector can be added to $\boldsymbol{x}_i$. A convenient choice is to identify the element of maximum magnitude in $\boldsymbol{u}_i$ and make the corresponding element in $\boldsymbol{x}_i$ equal to zero. Although other elements of $\boldsymbol{x}_i$ could be set to zero, this choice is most likely to produce a numerically well-conditioned problem. Substituting Eq. (7) into Eq. (6) gives

$$\left(s_i^2\boldsymbol{M} + s_i\boldsymbol{D} + \boldsymbol{K}\right)\boldsymbol{x}_i = \boldsymbol{F}_i\boldsymbol{x}_i = \boldsymbol{h}_i \tag{8}$$

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